University of Groningen
Exam Numerical Mathematics 1, June 13, 2016

Use of a simple calculator is allowed. All answers need to be motivated.
In front of the exercises you find its weight. In fact it gives the number of tenths which can be gained in the final mark. In total 5.4 points can be scored with this exam.

Exercise 1

(a) $\boxed{2}$ Polynomial interpolation of a set of data point (x_k, y_k), $k = 0, \cdots n$ can be written in the form of Lagrangian characteristic polynomials as

$$\Pi_n(x) = \sum_{k=0}^{n} y_k \varphi_k(x)$$

Give the general expression for $\varphi_k(x)$.

(b) For the numerical integration of $f(x)$ over the interval $[-1, 1]$ Simpson's rule is defined as

$$\frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1)$$

(i) $\boxed{2}$ Why is in general the sum of the weights of an integration rule equal to the length of the integration interval?

(ii) $\boxed{3}$ Show that the interpolating polynomial used to derive this method is of the function f at the point S=(-1 0 1)

$$\Pi_2(x) = \frac{1}{2}f(-1)x(x-1) + f(0)(1-x^2) + \frac{1}{2}f(1)x(x+1)$$

(iii) $\boxed{3}$ Derive Simpson's rule from the interpolating polynomial in the previous part.

(iv) $\boxed{3}$ The interval $[a, b]$ can be transformed to the interval $[-1, 1]$ by a linear function $x(s)$ such that $x(-1) = a$ and $x(1) = b$. Give $x(s)$ and use this to derive Simpson's rule for integration over $[a, b]$ from the formula given above.

Exercise 2

Consider the fixed point iteration $x^{(n+1)} = \Phi(x^{(n)})$ where $\Phi_1(x_1, x_2) = x_1 + \frac{1}{2}x_2$ and $\Phi_2(x_1, x_2) = \frac{1}{2}x_2 + \frac{1}{2}x_1 - \frac{1}{2}x_1^3$

(a) $\boxed{1}$ Show that the origin is a fixed point of this iteration.

(b) $\boxed{3}$ Compute the Jacobian matrix $J_\Phi(x)$ of $\Phi(x)$.

(c) $\boxed{4}$ Show that near the origin it also holds that $e^{(n+1)} \approx J_\Phi(0)e^{(n)}$ where $e^{(n)} = x^{(n)} - 0$.

(d) $\boxed{3}$ Using the eigenvalues of $J_\Phi(0)$ decide whether this fixed point method converges when starting close enough to the origin.

Continue on other side!

Exercise 3

(a) Consider the linear system $Ax = b$, where A and b are known and x has to be determined.

(i) $\boxed{4}$ Suppose that b is perturbed to $b + \Delta b$ resulting in a perturbed solution $x + \Delta x$. Show that

$$\frac{||\Delta x||}{||x||} \leq ||A||\ ||A^{-1}|| \frac{||\Delta b||}{||b||}$$

where the matrix and vector norm are related through

$$||A|| = \max_{x \neq 0} \frac{||Ax||}{||x||}$$

(ii) $\boxed{5}$ Let

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Consider iteration methods. Let P be the lower triangular part of A and define an iteration method by $Px^{(n+1)} = (P - A)x^{(n)} + b$. Show that the error $e^{(n)} = x^{(n)} - x$ satisfies the equation $e^{(n+1)} = Be^{(n)}$ where

$$B = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

(iii) $\boxed{3}$ Show that the method introduced in the previous part converges.

(iv) $\boxed{3}$ From the previous part we know that $e^{(n)} \to 0$ for $n \to \infty$. However, it will also converge quickly to a one dimensional subspace. Compute the generating vector of this space.

Exercise 4

Consider the elliptic problem

$$u - \frac{\partial}{\partial x}((1 + x)\frac{\partial u}{\partial x}) - \frac{\partial^2 u}{\partial y^2} = e^{x+y} \tag{1}$$

on the unit square, where on the boundaries we prescribe $u(x, 0) = 0,\ u(x, 1) = x,\ u(0, y) = 0,\ u(1, y) = y$.

(a) Show that for general $f(x)$ and $u(x)$, both sufficiently smooth,

$$\frac{d}{dx}(f(x)\frac{du(x)}{dx})_{x=x_m} = \frac{f(x_m + \frac{1}{2}h)(u(x_{m+1}) - u(x_m)) - f(x_m - \frac{1}{2}h)(u(x_m) - u(x_{m-1}))}{h^2} + O(h^2)$$

in three steps

(i) $\boxed{4}$ Show for general $w(x)$, smooth enough, that

$$\frac{dw}{dx}(x) = \frac{w(x + \frac{1}{2}h) - w(x - \frac{1}{2}h)}{h} - \frac{1}{24}h^2 \frac{d^3 w}{dx^3}(x) + O(h^3)$$

(ii) $\boxed{2}$ Show using the previous that

$$f(x_m + \frac{1}{2}h)\frac{du}{dx}(x_m + \frac{1}{2}h) = f(x_m + \frac{1}{2}h)\frac{u(x_{m+1}) - u(x_m)}{h} - \frac{1}{24}h^2 f(x_m + \frac{1}{2}h)\frac{d^3 u}{dx^3}(x_m + \frac{1}{2}h) + O(h^3)$$

(iii) $\boxed{3}$ Obtain the desired result by using the last two parts.

(b) Assume we use a mesh size $h = 1/(M + 1)$ in both directions.

(i) $\boxed{4}$ Give the discretization of (1) at a general grid point (m, n).

(ii) $\boxed{2}$ Give the discretization of the boundary conditions.